Comparative Statics and the Envelope Theorem

Econ 6105, Fall 2024

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(SB Chapters 15.1, 19.2)

Intro

We have worked through constrained maximization.
This is merely an input to most economic analyses
Economists usually focus on how the the solution *changes* as parameters change ("comparative static")

- Think about DWL from taxation.
- Think about interpretation of regression coefficients

Individual choices are impacted by a gazillion things, most of which are hard to measure

- Predicting the *levels* of people's choices is hard...

It is often more realistic (and helpful) to focus on how key economic factors (e.g.) price *change* the choices people make

- I.e. more concerned with $\frac{dx^*}{dp}$ than x^* itself.

Implicit Function Theorem

Consider a differentiable $G\left(x_1, ..., x_n, y^*(x_1, ..., x_n)\right) = c$.

- There may be some dependence of y^* on the xs
- E.g. Marginal cost (G) – which depends on input prices (xs) and the quantity choice made by the firm (y^*), which itself depends on prices – is equal to an output price (c)

If $\frac{\partial G}{\partial y^*} \neq 0$ when evaluated at $\mathbf{x}$, then at $\mathbf{x}$, differentiable $y^*(\mathbf{x})$ exists, and:

$$\frac{\partial y^*}{\partial x_i} = -\frac{\frac{\partial G}{\partial x_i}}{\frac{\partial G}{\partial y^*}} \tag{1}$$

(Basically just the Chain Rule.)
Even without knowing $y^*(\mathbf{x})$, we can say things about $\frac{\partial y^*}{\partial x_k}$, i.e. $\nabla y(\mathbf{x})$

Comparative Statics

Take the canonical 2-good consumer choice model: Maximize $u(x_1, x_2)$ s.t. $x_1 + p \cdot x_2 \leq I$.

Assume $u_1, u_2 > 0$ so the constraint binds. We know that at any local maximum, $\mathbf{x}^*$:

1. FOCs yield: $-p \cdot u_1(x_1^*, x_2^*) + u_2(x_1^*, x_2^*) = 0$
2. SOC implies: $p^2 \cdot u_{11}(x_1^*, x_2^*) - 2 \cdot p \cdot u_{12}(x_1^*, x_2^*) + u_{22}(x_1^*, x_2^*) < 0$

If we have suitable assumptions on $u(x_1, x_2)$ (e.g. concave), there is a local max, it is unique, and it is the global max.

Bread-and-butter of economics is "comparative statics"

- If I change an exogenous input (e.g. p), what happens to an endogenous output (e.g. x_2)?

Implicit Differentiation

Rewrite FOC:

$$-p \cdot u_1\Big(I - p \cdot x_2^*(p, I), x_2^*(p, I)\Big) + u_2\Big(I - p \cdot x_2^*(p, I), x_2^*(p, I)\Big) = 0 \quad (2)$$

Use the Chain Rule to differentiate the condition with respect to p:

Implicit Differentiation

Rewrite FOC:

$$-p \cdot u_1\Big(I - p \cdot x_2^*(p,I), x_2^*(p,I)\Big) + u_2\Big(I - p \cdot x_2^*(p,I), x_2^*(p,I)\Big) = 0 \quad (2)$$

Use the Chain Rule to differentiate the condition with respect to p:

$$-u_1 - p\cdot\Big(-u_{11}\cdot(x_2^* + p\cdot\frac{dx_2^*}{dp}) + u_{12}\cdot\frac{dx_2^*}{dp}\Big) + \Big(u_{12}\cdot -(x_2^* + p\cdot\frac{dx_2^*}{dp}) + u_{22}\cdot\frac{dx_2^*}{dp}\Big) = 0 \quad (3)$$

Solve for $\frac{dx_2^*}{dp}$:

$$\frac{dx_2^*}{dp} = \frac{u_1 - p \cdot x_2^* \cdot u_{11} + x_2^* \cdot u_{12}}{p^2 \cdot u_{11} - 2 \cdot p \cdot u_{12} + u_{22}} \quad (4)$$

Interpreting the Comparative Static

$$\frac{dx_2^*}{dp} = \frac{u_1 - p \cdot x_2^* \cdot u_{11} + x_2^* \cdot u_{12}}{p^2 \cdot u_{11} - 2 \cdot p \cdot u_{12} + u_{22}}$$

Interpreting the Comparative Static

$$\frac{dx_2^*}{dp} = \frac{u_1 - p \cdot x_2^* \cdot u_{11} + x_2^* \cdot u_{12}}{\underbrace{p^2 \cdot u_{11} - 2 \cdot p \cdot u_{12} + u_{22}}_{\text{SOC:}<0}}$$

So:

$$\frac{dx_2^*}{dp} \propto -u_1 + p \cdot x_2^* \cdot u_{11} - x_2^* \cdot u_{12}$$

Interpreting the Comparative Static

$$\frac{dx_2^*}{dp} = \frac{u_1 - p \cdot x_2^* \cdot u_{11} + x_2^* \cdot u_{12}}{\underbrace{p^2 \cdot u_{11} - 2 \cdot p \cdot u_{12} + u_{22}}_{\text{SOC:}<0}}$$

So:

$$\frac{dx_2^*}{dp} \propto \underbrace{-u_1}_{<0} + \underbrace{p \cdot x_2^* \cdot u_{11}}_{<0} \underbrace{-x_2^* \cdot u_{12}}_{?}$$

Sign is ambiguous. $p \uparrow \rightarrow x_2^* \downarrow$, unless $x_2^* \cdot u_{12}$ is very negative ("Giffen Good")

- If x_2^* is large, $p \uparrow$ causes a large fall in purchasing power.
- This could cause a large fall in x_1^*.
- If $u_{12} << 0$, i.e. the goods are strong substitutes, this could be enough for x_2^* to actually rise.

Ambiguous Results

Results where the sign is ambiguous are common

- If you only assume that u is an increasing, quasiconcave function, you are not being very specific, so your results likely won't be very "sharp"

But we actually learned a lot! The own-price elasticity depends crucially on:

- Substitutability with the other good
- The budget share of the good: if you're not buying much of the good in the first place, you will substitute away from it.

Parametric Example

If you assume $u(x_1, x_2) = x_1^{\alpha} \cdot x_2^{1-\alpha}$, then $x_2^* = (1-\alpha) \cdot I/p$. Therefore,

$$\frac{dx_2^*}{dp} = -\frac{(1-\alpha) \cdot I}{p^2} < 0$$

We now know the sign is negative.

- No surprise: $u_{12} = \frac{\alpha \cdot (1-\alpha)}{x_1^{1-\alpha} \cdot x_2^{\alpha}} > 0$, so previous result guarantees $\frac{dx_2^*}{dp} < 0$.

If you have data on I and p, you can estimate α and then do all sorts of analyses with great sharpness.

Generality vs. Sharpness

Specifying a utility function means making assumptions, whether you meant to or not!

- The result on the last slide that $\frac{dx_2^*}{dp} < 0$ was effectively an assumption, not a result
  - That utility function has $u_{12} > 0$, so $\frac{dx_2^*}{dp} < 0$ was guaranteed, even though we know it does not have to be true.

The sharper results come with a price: assumptions!

- That's not the end of the world, but you should be aware of which results are being driven by your assumptions

Start as general as you can; add assumptions as you need sharper results.

Choosing a Utility Function

When you take a model to the data, you'll need to make assumptions, among them choosing a functional form of $u()$

Key: assumptions should be "far" from the results you care about.

- "The own-price elasticity of apples is negative" cannot be tested with our utility function above
  - No dataset could ever reject that statement using that utility function
- "The own-price elasticity of apples is greater than oranges"' can be tested
  - Nothing in that utility function guarantees that apples will have the higher elasticity, or vice versa

We will not methodically go through different utility functions, but different fields – and research streams within those fields – have often settled on useful utility functions that simplify away unimportant things but leave the critical issues open to the data.

- E.g. CRRA vs. Epstein-Zin

Part of getting up to speed on a research stream is understanding what utility functions are used and why

Envelope Theorem

A very special type of comparative static looks at how a change in exogenous variable (e.g. price) changes the **objective function**

- For a consumer: utility
- For a firm: profit
- For a government: social welfare

Many important results in economics are a form of the Envelope Theorem. Define L to be the Lagrangian of a constrained optimization problem:

$$L = f(\mathbf{x}; a) + \lambda_1 \cdot (h_1(\mathbf{x}; a) - c_1) + ... + \lambda_M \cdot (h_M(\mathbf{x}; a) - c_M) \qquad (5)$$

Then $\frac{df(\mathbf{x}^*(a);a)}{da} = \frac{\partial L}{\partial a}$.

Translation in economics context: (assuming rational agents) we ignore behavioral response when doing the comparative static on welfare.

Utility Impact Example Without Envelope Theorem

Same example:

$$L = u(x_1, x_2) + \lambda \cdot (I - x_1 - p \cdot x_2)$$

Recall solution:

1. $\lambda^* = u_1 = u_2/p$
2. $x_1^* + p \cdot x_2^* = I$

Can differentiate $u(x_1^*(p), x_2^*(p))$ wrt p:

$$\begin{aligned}
\frac{du}{dp} &= u_1 \cdot \frac{dx_1}{dp} + u_2 \cdot \frac{dx_2}{dp} \\
&= \lambda^* \cdot \frac{dx_1}{dp} + p \cdot \lambda^* \cdot \frac{dx_2}{dp} \text{ (by FOC)} \\
&= \lambda^* \cdot (\frac{dx_1}{dp} + p \cdot \frac{dx_2}{dp}) \\
&= -\lambda^* \cdot x_2^* \text{ (by budget constraint + product rule)}
\end{aligned} \tag{6}$$

Utility Impact Example With Envelope Theorem

$$L = u(x_1, x_2) + \lambda \cdot (I - x_1 - p \cdot x_2)$$

Can take partial derivative of L wrt p:

$$\frac{\partial L}{\partial p} = -\lambda^* \cdot x_2^* \tag{7}$$

Jumped to the same answer without using the FOC or differentiating the budget constraint!

Envelope Theorem Intuition

The Envelope Theorem holds because the optimizing agent is indifferent to small perturbations of the bundle

- FOC ensures this: $$\underbrace{u_1}_{\text{value of \$1 spent on } x_1} = \underbrace{u_2/p}_{\text{value of \$1 spent on } x_2}$$

So the behavioral response has no (first-order) impact on utility

First-order impact on welfare is simple:

- Higher price leads to lower purchasing power in the amount of $dp \cdot x_2^*$
- This lowers utility by the marginal utility of income (λ^*) per dollar
- Change in utility per unit price change is $-\lambda^* \cdot x_2^*$
  - Behavioral response netted out, because of optimality (i.e. FOC)

Example: Marginal Deadweight Loss of a Tax

Consider a government that taxes x_2. Government's objective function is:

$$W(t) = \mu \cdot t \cdot x_2^* + \max_{x_1, x_2} \left\{ u(x_1, x_2) + \lambda \cdot (I - x_1 - (p + t) \cdot x_2) \right\} \quad (8)$$

Translation:

- Government raises revenue $(t \cdot x_2^*)$, which it values at μ per dollar
- It also cares about the consumer's welfare, $u(x_1^*, x_2^*)$
- It also understands that the tax will impact the consumer's behavior.

2 simplifying assumptions (not necessary, but clean things up):

1. Supply of goods is perfectly elastic (i.e. p is constant)
2. $\mu = \lambda$: government values a dollar at the same rate that a consumer does. Natural, but can certainly consider alternatives.

Calculating $dW(t)/dt$ Without Envelope Theorem

$$W(t) = \lambda \cdot t \cdot x_2^* + \max_{x_1,x_2} \left\{ u(x_1,x_2) + \lambda \cdot (I - x_1 - (p+t)\cdot x_2) \right\} \tag{9}$$

$$\frac{dW(t)}{dt} = \lambda \cdot (x_2^* + t \cdot \frac{dx_2^*}{dt}) + u_1 \cdot \frac{dx_1^*}{dt} + u_2 \cdot \frac{dx_2^*}{dt} - \lambda \cdot (\frac{dx_1^*}{dt} + (p+t) \cdot \frac{dx_2^*}{dt} + x_2^*)$$

$$= \lambda \cdot (t \cdot \frac{dx_2^*}{dt} - \frac{dx_1^*}{dt} - (p+t) \cdot \frac{dx_2^*}{dt}) + u_1 \cdot \frac{dx_1^*}{dt} + u_2 \cdot \frac{dx_2^*}{dt}$$

$$= \lambda \cdot (t \cdot \frac{dx_2^*}{dt} - \frac{dx_1^*}{dt} - (p+t) \cdot \frac{dx_2^*}{dt}) + \lambda \cdot \frac{dx_1^*}{dt} + \lambda \cdot (p+t) \cdot \frac{dx_2^*}{dt} \text{ (by FOC)}$$

$$= \lambda \cdot t \cdot \frac{dx_2^*}{dt}$$

Calculating $dW(t)/dt$ With Envelope Theorem

$$W(t) = \lambda \cdot t \cdot x_2^* + \max_{x_1, x_2} \left\{ u(x_1, x_2) + \lambda \cdot (I - x_1 - (p + t) \cdot x_2) \right\}$$

$$\begin{aligned}
\frac{dW(t)}{dt} &= \lambda \cdot (x_2^* + t \cdot \frac{dx_2^*}{dt}) + \frac{\partial}{\partial t} \Big\{ u(x_1, x_2) + \lambda \cdot (I - x_1 - (p + t) \cdot x_2) \Big\} \\
&= \lambda \cdot (x_2^* + t \cdot \frac{dx_2^*}{dt}) - \lambda \cdot x_2^* \\
&= \lambda \cdot t \cdot \frac{dx_2^*}{dt}
\end{aligned}$$

Deadweight Loss: Triangles and Rectangles

Dollar value of marginal deadweight loss of taxation is $\frac{dW/dt}{\lambda} = t \cdot \frac{dx_2}{dt}$.
When you tax, 3 things occur:

1. Mechanical transfer (i.e. ignoring behavioral response) from consumer to government
   - This cancels out $(+\lambda \cdot x_2^* - \lambda \cdot x_2^*)$
2. Behavioral response has a direct impact on consumer welfare
   - Envelope Theorem tells us this is zero!
3. Behavioral response lowers government revenue ("fiscal externality")
   - That's the term we see: $t \cdot \frac{dx_2}{dt}$

Taught graphically to students in Econ 101:

- When $t = 0$ there is no (first-order) deadweight loss ("triangle")
- When $t > 0$, the behavioral response leads to some lost revenue, which is a first-order loss ("rectangle")

Sufficient Statistics

Found that marginal DWL is just a function of the level of the tax (t) and the behavioral response ($\frac{dx_2}{dt}$)

- Both of these are (in principle) measurable in data

Note how powerful this result is: we got a very important output (DWL of taxation) while making no assumptions on the utility function.

- Using earlier terms, we got a very sharp result without paying a heavy cost in assumptions

We have still (implicitly) assumed the consumer makes the optimal decision. What if we relaxed that assumption?

- E.g. What if, for some reason, the consumer starts off consuming too much x_1 and not enough x_2?

Or what if you are studying a non-marginal change in a tax?