Understanding "Negative Definite:" Some Matrix Algebra

Econ 6105, Fall 2024

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(SB Chapters 8.1, 9.1, 16.2)

Introduction

We are building towards optimization.
Conclusion of previous slide deck:

- "Negative definite" "Hessian matrix" is key for finding a local max

This deck introduces the bare minimum of linear algebra to make sense of that statement

- Later, we will return to more advanced linear algebra

Scalars, Vectors, and Matrices

A scalar is a real number

- E.g. 4.4

A vector of length n is a collection of n scalars

- E.g $\begin{bmatrix} 2.1 \\ 1.87 \\ 4 \end{bmatrix}$
- A scalar is a vector of length 1

A matrix is a rectangular array of scalars, with k rows and n columns

- E.g $\begin{bmatrix} 2.1 & 3.0 \\ 1.87 & 4.4 \\ -0.3 & 5 \end{bmatrix}$
- A vector is a matrix with 1 column

Transpose

Consider a *kxn* matrix A.
The "transpose" of A, denoted A^T is *nxk* matrix such that the i^{th} *row* of A is the i^{th} *column* of A^T.

- E.g. $A = \begin{bmatrix} 2 & 4 & -1 \\ -2 & 0 & 0.1 \end{bmatrix}$; $A^T = \begin{bmatrix} 2 & -2 \\ 4 & 0 \\ -1 & 0.1 \end{bmatrix}$

Addition and Scalar Multiplication

For a scalar c and matrix A, find $B = c \cdot A$ with $b_{ij} = c \cdot a_{ij}$.

- E.g. $4 \cdot \begin{bmatrix} 3 & 2 \\ 4 & 1 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 12 & 8 \\ 16 & 4 \\ 0 & 24 \end{bmatrix}$

For a matrices A and B, find $C = A + B$ with $c_{ij} = a_{ij} + b_{ij}$.

- E.g. $\begin{bmatrix} 12 & 8 \\ 16 & 4 \\ 0 & 24 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ 4 & 1 \\ 0 & 6 \end{bmatrix} = \begin{bmatrix} 15 & 10 \\ 20 & 5 \\ 0 & 30 \end{bmatrix}$

Dot product

While addition is straightforward, matrix multiplication is odd

Useful to first consider the "dot product" of two vectors.

For 2 vectors of length n, their dot product is:

$$v \cdot w = \begin{bmatrix} v_1 \\ v_2 \\ \dots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ \dots \\ w_n \end{bmatrix} = \sum_{i=1}^{n} v_i \cdot w_i$$

- E.g. $\begin{bmatrix} 4 \\ 3 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = 4 \cdot 0 + 3 \cdot 1 + -3 \cdot 2 = -3$

Matrix Multiplication

For matrices A and B, the product $C = AB$ is only defined if the number of *columns of A* equals the number of *rows of B*.
In that case, c_{ij} is found as the dot product of the row i of A with column j of B.

- E.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

Matrix Multiplication

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- E.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot 1 + 2 \cdot 3 + 3 \cdot 5 & 1 \cdot 2 + 2 \cdot 4 + 3 \cdot 6 \\ 4 \cdot 1 + 5 \cdot 3 + 6 \cdot 5 & 4 \cdot 2 + 5 \cdot 4 + 6 \cdot 6 \end{bmatrix}$

$= \begin{bmatrix} 22 & 28 \\ 49 & 64 \end{bmatrix}$

Resulting product has A number of *rows* and B's number of *columns*

$AB \neq BA$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$AB \neq BA$

$$\begin{bmatrix}1 & 2\\3 & 4\\5 & 6\end{bmatrix}\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix} = \begin{bmatrix}1\cdot1+2\cdot4 & 1\cdot2+2\cdot5 & 1\cdot3+2\cdot6\\3\cdot1+4\cdot4 & 3\cdot2+4\cdot5 & 3\cdot3+4\cdot6\\5\cdot1+6\cdot4 & 5\cdot2+6\cdot5 & 5\cdot3+6\cdot6\end{bmatrix}$$

$$= \begin{bmatrix}9 & 12 & 15\\19 & 26 & 33\\29 & 40 & 51\end{bmatrix}$$

AB was a 2x2 matrix, but BA is a 3x3 matrix.

Other Rules of Matrix Algebra

While matrices do not obey commutativity of multiplication, they obey other standard rules:

- Associativity:
  - Addition: $(A + B) + C = A + (B + C)$
  - Multiplication: $(AB)C = A(BC)$
- Commutativity of Addition: $A + B = B + A$

Determinant of "small" Square Matrices

A square matrix has the same number of rows as columns (nxn).
A square matrix's "determinant" has many different uses, which we will use later. For "small" matrices, the formulas are fairly simple:

- $A = [a_{11}]$, $det(A) = a_{11}$
- $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $det(A) = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$
- $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $det(A) =$

  $$a_{11} \cdot det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \cdot det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \cdot det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

General Formula for a Determinant

Consider a nxn matrix A.

- Define A_{ij} to be a $(n-1)x(n-1)$ matrix created by removing row i and column j from A.
- Define $M_{ij} = det(A_{ij})$.

$det(A) = a_{11} \cdot M_{11} - a_{12} \cdot M_{12} + ... + (-1)^{n+1} \cdot a_{1n} \cdot M_{1n}$

Negative Definite Matrices

We say a scalar c is "negative" if $c < 0$.
It is less straightforward what a "negative" matrix should be.
A close analog for square matrices is that a nxn matrix A is "negative definite" if $\forall \mathbf{x} \neq \mathbf{0} \in R^n$:

$$\mathbf{x}^T A \mathbf{x} < 0. \tag{1}$$

Note that this is sensible in the intuitive case of $n = 1$ (scalars).

- It amounts to saying A is negative if $x^2 \cdot a < 0$ for all x. Since $x^2 > 0$, this is what we would expect.
- The definition above extends to matrices with $n > 1$
- Unlike scalars, though, a matrix might not be negative (definite) or positive (definite)...it could be "indefinite."

Negative Definiteness of Symmetric 2x2 Matrices

Consider a symmetric 2x2 matrix: $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$.

For a generic $\mathbf{x} \in R^2$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, we have:

$$\mathbf{x}^T A \mathbf{x} =$$

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$$\mathbf{x}^T A\mathbf{x} = \begin{bmatrix} a \cdot x_1 + b \cdot x_2 & b \cdot x_1 + c \cdot x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$$

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$$= a \cdot x_1^2 + 2 \cdot b \cdot x_2 \cdot x_1 + c \cdot x_2^2$$

Negative Definiteness of Symmetric 2x2 Matrices (2)

$$\begin{aligned}
\mathbf{x}^T A\mathbf{x} &= a\cdot x_1^2 + 2\cdot b\cdot x_2\cdot x_1 + c\cdot x_2^2 \\
&= a\cdot x_1^2 + 2\cdot b\cdot x_2\cdot x_1 + c\cdot x_2^2 + \frac{b^2}{a}\cdot x_2^2 - \frac{b^2}{a}\cdot x_2^2 \\
&= a\cdot\left(x_1^2 + \frac{2\cdot b}{a}\cdot x_2\cdot x_1 + \frac{b^2}{a^2}\cdot x_2^2\right) + c\cdot x_2^2 - \frac{b^2}{a}\cdot x_2^2 \\
&= a\cdot\left(x_1 + \frac{b}{a}\cdot x_2\right)^2 + \frac{a\cdot c - b^2}{a}\cdot x_2^2
\end{aligned}$$

Negative Definiteness of Symmetric 2x2 Matrices (2)

$$
\begin{aligned}
\mathbf{x}^T A\mathbf{x} &= a \cdot x_1^2 + 2 \cdot b \cdot x_2 \cdot x_1 + c \cdot x_2^2 \\
&= a \cdot x_1^2 + 2 \cdot b \cdot x_2 \cdot x_1 + c \cdot x_2^2 + \frac{b^2}{a} \cdot x_2^2 - \frac{b^2}{a} \cdot x_2^2 \\
&= a \cdot \left(x_1^2 + \frac{2 \cdot b}{a} \cdot x_2 \cdot x_1 + \frac{b^2}{a^2} \cdot x_2^2\right) + c \cdot x_2^2 - \frac{b^2}{a} \cdot x_2^2 \\
&= a \cdot \underbrace{\left(x_1 + \frac{b}{a} \cdot x_2\right)^2}_{+} + \frac{a \cdot c - b^2}{a} \cdot \underbrace{x_2^2}_{+}
\end{aligned}
$$

Will be negative for all x_1, x_2 if and only if:

1. $a < 0$
2. $a \cdot c - b^2 > 0$

Negative Definiteness of Symmetric 2x2 Matrices (2)

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\begin{aligned}
\mathbf{x}^T A \mathbf{x} &= a \cdot x_1^2 + 2 \cdot b \cdot x_2 \cdot x_1 + c \cdot x_2^2 \\
&= a \cdot x_1^2 + 2 \cdot b \cdot x_2 \cdot x_1 + c \cdot x_2^2 + \frac{b^2}{a} \cdot x_2^2 - \frac{b^2}{a} \cdot x_2^2 \\
&= a \cdot \left(x_1^2 + \frac{2 \cdot b}{a} \cdot x_2 \cdot x_1 + \frac{b^2}{a^2} \cdot x_2^2 \right) + c \cdot x_2^2 - \frac{b^2}{a} \cdot x_2^2 \\
&= a \cdot \underbrace{\left(x_1 + \frac{b}{a} \cdot x_2 \right)^2}_{+} + \frac{a \cdot c - b^2}{a} \cdot \underbrace{x_2^2}_{+}
\end{aligned}
$$

Will be negative for all x_1, x_2 if and only if:

1. $a < 0$, i.e. $a_{11} < 0$
2. $a \cdot c - b^2 > 0$, i.e. $det(A) > 0$

Negative Definiteness of General Symmetric Matrices

Consider symmetric *nxn* matrix $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \dots & & & \\ a_{1n} & a_{2n} & \dots & a_{nn} \end{bmatrix}$.

Define the "first principal submatrix" as $A_1 \equiv [a_{11}]$.

Define the "second principal submatrix" as $A_2 \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$.

Similarly define the k^{th} principal submatrix of A by deleting the last $n-k$ rows and columns of A.

SB Theorem 16.1: A symmetric matrix A as defined above is negative definite if and only if

1. $det(A_1) < 0$
2. $det(A_2) > 0$
3. $det(A_3) < 0$
4. $...det(A_k) \propto (-1)^k$

Positive Definite Symmetric Matrices

A *nxn* matrix A is “positive definite” if $\forall \mathbf{x} \neq \mathbf{0} \in R^n$:

$$\mathbf{x}^T A \mathbf{x} > 0. \tag{2}$$

SB Theorem 16.1: A symmetric matrix A as defined on the previous slide is positive definite if and only if:

- $det(A_k) > 0 \; \forall k < n$.

THE BIG IDEA

End of last slide deck:

Extrema of Multivariate Functions

As with single-variable functions, a "derivative" of 0 indicates a local extremum:

$$\nabla f(\mathbf{x}) = \mathbf{0}$$

i.e. $\frac{\partial f}{\partial x_i} = 0 \; \forall i$.
Again, in general need to worry about multiplicity of local extrema, boundaries, strange points.
But with an concave function, a local maximum will be unique, and it will be the global maximum.
A multivariate function is concave if its Hessian is "negative definite."

- We'll now need to discuss some linear algebra.

THE BIG IDEA

Consider $\mathbf{x}$ such that:

1. $\nabla f(\mathbf{x}) = \mathbf{0}$ (i.e. derivative is 0)
2. $H(\mathbf{x})$ is negative definite

If we perturb $\mathbf{x}$ by $\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_1 \\ \dots \\ \epsilon_n \end{bmatrix}$, second-order Taylor approximation yields:

$$f(\mathbf{x} + \boldsymbol{\epsilon}) - f(\mathbf{x}) \approx \underbrace{\nabla f(\mathbf{x}) \cdot \boldsymbol{\epsilon}}_{\text{0: critical point}} + \underbrace{\frac{1}{2} \cdot \boldsymbol{\epsilon}^T H(\mathbf{x}) \boldsymbol{\epsilon}}_{\text{<0: neg. def. Hessian}}$$

So we have justified that if we are at a critical point with a negative definite Hessian, we are at a local max: no matter which directions you move, f falls!
By the same logic as before, if the Hessian is negative definite for all points in the domain, the local max is a global max.

A Little More Intuition for the 2x2 Case

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{bmatrix}$$

H being negative definite implies 3 things:

1. $f_{11} < 0$
   - Intuitive based on single-variable intuition
2. $f_{22} < 0$
   - Intuitive based on single-variable intuition
3. $f_{11} \cdot f_{22} > f_{12}^2$
   - The negatives from the own-seconds outweigh any positives that could come from the cross-partials